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THE CONDITIONS FOR A PLAIT POINT

BY PAUL SAUREL

IF a plane, tangent to a surface in two points, moves in such a way that the two points of contact tend to coalesce, the limiting position of these points of contact is called a plait point.* This definition applies not only to a surface in space of three dimensions but also to a hypersurface in a space of n dimensions.

Let us suppose that the equation of a surface in a space of $n+1$ dimensions is

$$x = f(x_1, x_2, \cdot \cdot \cdot, x_n), \tag{1}$$

and let us write

$$\Delta = \begin{vmatrix} f_{11}, f_{12}, & \cdot \cdot \cdot & , f_{1n} \\ f_{21}, f_{22}, & \cdot \cdot \cdot & , f_{2n} \\ \cdot \quad \cdot & & \cdot \\ \cdot \quad \cdot & & \cdot \\ \cdot \quad \cdot & & \cdot \\ f_{n1}, f_{n2}, & \cdot \cdot \cdot & , f_{nn} \end{vmatrix}, \tag{2}$$

in which

$$f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \tag{3}$$

At a plait point of the surface, every determinant of order n that can be formed from the matrix

$$\begin{array}{cccc} f_{11}, & f_{12}, & \cdot \cdot \cdot & , f_{1n} \\ f_{21}, & f_{22}, & \cdot \cdot \cdot & , f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1}, & f_{n2}, & \cdot \cdot \cdot & , f_{nn} \\ \frac{\partial \Delta}{\partial x_1}, & \frac{\partial \Delta}{\partial x_2}, & \cdot \cdot \cdot \cdot & , \frac{\partial \Delta}{\partial x_n} \end{array} \tag{4}$$

reduces to zero.

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* Korteweg, *Archives Néerlandaises des sciences exactes et naturelles*, vol. 24, p. 57 (1890).

These conditions are due to Gibbs.* The reasoning by which he establishes them is, however, stated in physical terms. It may therefore be well to divest his argument of its physical garb and to present it, or rather an equivalent of it, in mathematical terms.

The equations which express the coincidence of the tangent planes drawn to the surface 1 at two points $x, x_1, x_2, \cdots, x_n$ and $x', x'_1, x'_2, \cdots, x'_n$ may be written

$$f'_i = f_i \qquad i = 1, 2, \cdots, n, \tag{5}$$

$$f' - \sum_{i=1}^{n} x'_i f'_i = f - \sum_{i=1}^{n} x_i f_i, \tag{6}$$

in which

$$f_i = \frac{\partial f}{\partial x_i}, \tag{7}$$

and the primed letters indicate quantities connected with the point $x', x'_1, x'_2, \cdots, x'_n$.

If we suppose the two points of contact to be very near together, these equations may be written

$$\sum_{j=1}^{n} (x'_j - x_j) f_{ij} + \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (x'_j - x_j)(x'_k - x_k) f_{ijk} + \cdots = 0, \qquad i = 1, 2, \cdots, n, \tag{8}$$

$$\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x'_i - x_i)(x'_j - x_j) f_{ij} + \frac{1}{2.3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (x'_i - x_i)(x'_j - x_j)(x'_k - x_k) f_{ijk} + \cdots = 0, \tag{9}$$

in which

$$f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}. \tag{10}$$

The omitted terms are of the third and fourth orders respectively.

Multiplying equations 8 by $x'_1 - x_1, x'_2 - x_2, \cdots, x'_n - x_n$ and subtracting one half their sum from equation 9, we find that this equation can be replaced by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (x'_i - x_i)(x'_j - x_j)(x'_k - x_k) f_{ijk} + \cdots = 0. \tag{11}$$

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* On the Equilibrium of Heterogeneous Substances, *Transactions of the Connecticut Academy of Arts and Sciences*, vol. 3, part 1, p. 191 (1876).

The conditions for a plait point are thus, first, that the equations

$$\sum_{j=1}^{n}(x_j' - x_j)f_{ij} = 0, \qquad i = 1, 2, \cdots, n, \tag{12}$$

should be compatible, which requires that

$$\Delta = 0;$$

and, second, that the values of $x_j' - x_j$ which satisfy equations 12 should also satisfy the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_i' - x_i)(x_j' - x_j)(x_k' - x_k)f_{ijk} = 0. \tag{14}$$

If we denote the minor of Δ which corresponds to f_{ij} by F_{ij}, the values of $x_1' - x_1, x_2' - x_2, \cdots, x_n' - x_n$ which satisfy equations 12 are given by any one of the n sets of equations

$$x_1' - x_1 : x_2' - x_2 : \cdots : x_n' - x_n = F_{r1} : F_{r2} : \cdots : F_{rn}, \qquad r = 1, 2, \cdots, n. \tag{15}$$

Now, equation 14 can be written in the form

$$\sum_{k=1}^{n}(x_k' - x_k)\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i' - x_i)(x_j' - x_j)f_{ijk} = 0. \tag{16}$$

If we compare this equation with equations 12 it follows that every determinant of order n formed from the matrix

$$\begin{matrix} f_{11}, & f_{12}, & \cdots, & f_{1n} \\ f_{21}, & f_{22}, & \cdots, & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1}, & f_{n2}, & \cdots, & f_{nn} \\ A_1, & A_2, & \cdots, & A_n, \end{matrix} \tag{17}$$

where $$A_k = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i' - x_i)(x_j' - x_j)f_{ijk},$$

must be equal to zero.

We shall now show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i' - x_i)(x_j' - x_j)f_{ijr} \tag{18}$$

is proportional to

$$\frac{\partial\Delta}{\partial x_r},$$

so that the matrix 17 is equivalent to the matrix 4. We shall thus have established Gibbs's conditions.

From equations 15 it follows that 18 is proportional to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ijr} F_{si} F_{sj} . \tag{19}$$

On the other hand

$$\frac{\partial \Delta}{\partial x_r} = \begin{vmatrix} f_{11r}, f_{12r}, \cdots, f_{1nr} \\ f_{21}, f_{22}, \cdots, f_{2n} \\ \cdot \quad \cdot \qquad \cdot \\ \cdot \quad \cdot \qquad \cdot \\ \cdot \quad \cdot \qquad \cdot \\ f_{n1}, f_{n2}, \cdots, f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11}, f_{12}, \cdots, f_{1n} \\ f_{21r}, f_{22r}, \cdots, f_{2nr} \\ \cdot \quad \cdot \qquad \cdot \\ \cdot \quad \cdot \qquad \cdot \\ \cdot \quad \cdot \qquad \cdot \\ f_{n1}, f_{n2}, \cdots, f_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} f_{11}, f_{12}, \cdots, f_{1n} \\ f_{21}, f_{22}, \cdots, f_{2n} \\ \cdot \quad \cdot \qquad \cdot \\ \cdot \quad \cdot \qquad \cdot \\ \cdot \quad \cdot \qquad \cdot \\ f_{n1r}, f_{n2r}, \cdots, f_{nnr} \end{vmatrix}$$

$$\begin{aligned} &= f_{11r} F_{11} + f_{12r} F_{12} + \cdots + f_{1nr} F_{1n} \\ &+ f_{21r} F_{21} + f_{22r} F_{22} + \cdots + f_{2nr} F_{2n} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ f_{n1r} F_{n1} + f_{n2r} F_{n2} + \cdots + f_{nnr} F_{nn} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ijr} F_{ij}. \end{aligned} \tag{20}$$

From this equation we get

$$F_{ss} \frac{\partial \Delta}{\partial x_r} = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ijr} F_{ss} F_{ij}. \tag{21}$$

Now since $\Delta = 0$, it follows from an elementary theorem in determinants that

$$F_{ss} F_{ij} - F_{si} F_{sj} = 0. \tag{22}$$

A comparison of 18, 19, 21, and 22 then shows immediately that expression 18 is proportional to $\frac{\partial \Delta}{\partial x_r}$.

The theorem is thus established.

The above conclusion fails if $F_{ss} = 0$ for every value of s. But, in that case, since

$$F_{ss} F_{rr} = F^2_{rs},$$

it follows that, for all values of i and j, $F_{ij} = 0$. The equations 12 no longer uniquely determine the ratios of $x_1' - x_1$, $x_2' - x_2$, $\cdots$, $x_n' - x_n$ and the singularity is of a more complicated character.

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The demonstration given in my note "On a Theorem of Kinematics" in the ANNALS OF MATHEMATICS, second series, vol. 2, p. 159, is, in substance, the same as that given by Kelvin and Tait in their *Treatise on Natural Philosophy*, vol. 1, p. 78. I regret that I did not become aware of the fact sooner.

PAUL SAUREL.

NOVEMBER, 1903.